COMPATIBILITY OF A NONCOMMUTATIVE PROBABILITY SPACE AND A NONCOMMUTATIVE PROBABILITY SPACE WITH AMALGAMATION AND ITS APPLICATION

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Abstract. In this paper, we will consider R-transform theory and R-transform calculus for compatible noncommutative probability space and amagamated noncommutative probability space (See [18]). By doing this, we can realize the relation between scalar-valued R-transforms and operator-valued moment series, under the compatibility.

Voiculescu developed Free Probability Theory. Here, the classical concept of Independence in Probability theory is replaced by a noncommutative analogue called Freeness (See [20]). There are two approaches to study Free Probability Theory. One of them is the original analytic approach of Voiculescu and the other one is the combinatorial approach of Speicher and Nica (See [19], [1] and [17]).

To observe the free additive convolution and free multiplicative convolution of two distributions induced by free random variables in a noncommutative probability space (over $B = \mathbb{C}$), Voiculescu defined the R-transform and the S-transform, respectively. These show that to study distributions is to study certain (B-)formal series for arbitrary noncommutative indeterminants.

Speicher defined the free cumulants which are the main objects in Combinatorial approach of Free Probability Theory. And he developed free probability theory by using Combinatorics and Lattice theory on collections of noncrossing partitions (See [17]). Also, Speicher considered the operator-valued free probability theory, which is also defined and observed analytically by Voiculescu, when $\mathbb{C}$ is replaced to an arbitrary algebra B (See [19]). Nica defined R-transforms of several random variables (See [1]). He defined these R-transforms as multivariable formal series in noncommutative several indeterminants. To observe the R-transform, the Möbius Inversion under the embedding of lattices plays a key role (See [19],[17],[5],[20],[20], [17], [31] and [32]).

In [20], [19] and [20], we observed the amalgamated R-transform calculus. Actually, amalgamated R-transforms are defined originally by Voiculescu (See [11]) and are characterized combinatorially by Speicher (See [19]). In [20], we defined amalgamated R-transforms slightly differently from those defined in [19] and [11]. We defined them as B-formal series and tried to characterize, like in [1] and [17]. The

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main tool which is considered, for studying amalgamated R-transforms is amalgamated boxed convolution. After defining boxed convolution over an arbitrary algebra B, we could get that

$$R_{x_1,...,x_s} \boxed{*} R^{symm(1_B)}_{y_1,...,y_s} = R_{x_1y_1,...,x_sy_s}, \text{ for any } s \in \mathbb{N},$$

where x_j's and y_j's are free B-valued random variables.

In this paper, we will consider the relation between a noncommutative probability space and a noncommutative probability space with amalgamation over an arbitrary unital algebra. Let B be a unital algebra and let A be an algebra over B. If $E : A \rightarrow B$ is the conditional expectation, then we have the noncommutative probability space over B, (A, E). Now, let $\varphi : A \rightarrow \mathbb{C}$ be a linear functional. Then we have the (scalar-valued) noncommutative probability space (A, φ). We say that the spaces (A, E) and (A, φ) are compatible if $\varphi(x) = \varphi\left(E(x)\right)$, for all $x \in A$. Under compatibility, we will consider the scalar-valued R-transform theory and amalgamated R-transform theory.

1. Preliminaries

1.1. **Amalgamated Free Probability Theory.**

In this section, we will summarize and introduced the basic results from [19] and [20]. Throughout this section, let B be a unital algebra. The algebraic pair (A, φ) is said to be a noncommutative probability space over B (shortly, NCPSpace over B) if A is an algebra over B (i.e $1_B = 1_A \in B \subset A$) and $\varphi : A \rightarrow B$ is a B-functional (or a conditional expectation) ; φ satisfies

$$\varphi(b) = b, \text{ for all } b \in B$$

and

$$\varphi(bxb') = b\varphi(x)b', \text{ for all } b, b' \in B \text{ and } x \in A.$$

Let (A, φ) be a NCPSpace over B. Then, for the given B-functional, we can determine a moment multiplicative function $\widehat{\varphi} = (\varphi^{(n)})_{n=1}^{\infty} \in I(A, B)$, where

$$\varphi^{(n)}(a_1 \otimes ... \otimes a_n) = \varphi(a_1....a_n),$$

for all $a_1 \otimes ... \otimes a_n \in A^{\otimes_B n}$, $\forall n \in \mathbb{N}$.

We will denote noncrossing partitions over $\{1, ..., n\}$ ($n \in \mathbb{N}$) by $NC(n)$. Define an ordering on $NC(n)$;

$\theta = \{V_1, ..., V_k\} \leq \pi = \{W_1, ..., W_l\} \overset{def}{\Leftrightarrow}$ For each block $V_j \in \theta$, there exists only one block $W_p \in \pi$ such that $V_j \subset W_p$, for $j = 1, ..., k$ and $p = 1, ..., l$.

Then $(NC(n), \leq)$ is a complete lattice with its minimal element $0_n = \{(1), ..., (n)\}$ and its maximal element $1_n = \{(1, ..., n)\}$. We define the incidence algebra I_2 by a set of all complex-valued functions η on $\cup_{n=1}^{\infty} (NC(n) \times NC(n))$ satisfying $\eta(\theta, \pi) = 0$, whenever $\theta \nleq \pi$. Then, under the convolution

$$* : I_2 \times I_2 \to \mathbb{C}$$

defined by

$$\eta_1 * \eta_2(\theta, \pi) = \sum_{\theta \leq \sigma \leq \pi} \eta_1(\theta, \sigma) \cdot \eta_2(\sigma, \pi),$$

I_2 is indeed an algebra of complex-valued functions. Denote zeta, Möbius and delta functions in the incidence algebra I_2 by ζ, μ and δ, respectively. i.e

$$\zeta(\theta, \pi) = \begin{cases} 1 & \theta \leq \pi \\ 0 & otherwise, \end{cases}$$

$$\delta(\theta, \pi) = \begin{cases} 1 & \theta = \pi \\ 0 & otherwise, \end{cases}$$

and μ is the $(*)$-inverse of ζ. Notice that δ is the $(*)$-identity of I_2. By using the same notation $(*)$, we can define a convolution between $I(A, B)$ and I_2 by

$$\widehat{f} * \eta \,(a_1, ..., a_n\,;\, \pi) = \sum_{\pi \in NC(n)} \widehat{f}(\pi)(a_1 \otimes ... \otimes a_n)\eta(\pi, 1_n),$$

where $\widehat{f} \in I(A, B)$, $\eta \in I_2$, $\pi \in NC(n)$ and $a_j \in A$ ($j = 1, ..., n$), for all $n \in \mathbb{N}$. Notice that $\widehat{f} * \eta \in I(A, B)$, too. Let $\widehat{\varphi}$ be a moment multiplicative function in $I(A, B)$ which we determined before. Then we can naturally define a cumulant multiplicative function $\widehat{c} = (c^{(n)})_{n=1}^{\infty} \in I(A, B)$ by

$$\widehat{c} = \widehat{\varphi} * \mu \quad \text{ or } \quad \widehat{\varphi} = \widehat{c} * \zeta.$$

This says that if we have a moment multiplicative function, then we always get a cumulant multiplicative function and vice versa, by $(*)$. This relation is so-called "Möbius Inversion". More precisely, we have

$$\begin{aligned}\varphi(a_1...a_n) &= \varphi^{(n)}(a_1 \otimes ... \otimes a_n) \\ &= \sum_{\pi \in NC(n)} \widehat{c}(\pi)(a_1 \otimes ... \otimes a_n)\zeta(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \widehat{c}(\pi)(a_1 \otimes ... \otimes a_n),\end{aligned}$$

for all $a_j \in A$ and $n \in \mathbb{N}$. Or equivalently,

$$c^{(n)}(a_1 \otimes ... \otimes a_n) = \sum_{\pi \in NC(n)} \widehat{\varphi}(\pi)(a_1 \otimes ... \otimes a_n)\mu(\pi, 1_n).$$

Now, let (A_i, φ_i) be NCPSpaces over B, for all $i \in I$. Then we can define a amalgamated free product of A_i 's and amalgamated free product of φ_i's by

$$A \equiv *_B A_i \quad \text{and} \quad \varphi \equiv *_i \varphi_i,$$

respectively. Then, by Voiculescu, (A, φ) is again a NCPSpace over B and, as a vector space, A can be represented by

$$A = B \oplus \left(\oplus_{n=1}^{\infty} \left(\underset{i_1 \neq ... \neq i_n}{\oplus} (A_{i_1} \ominus B) \otimes ... \otimes (A_{i_n} \ominus B) \right) \right),$$

where $A_{i_j} \ominus B = \ker \varphi_{i_j}$. We will use Speicher's combinatorial definition of amalgamated free product of B-functionals ;

Definition 1.1. *Let (A_i, φ_i) be NCPSpaces over B, for all $i \in I$. Then $\varphi = *_i \varphi_i$ is the amalgamated free product of B-functionals φ_i's on $A = *_B A_i$ if the cumulant multiplicative function $\widehat{c} = \widehat{\varphi} * \mu \in I(A, B)$ has its restriction to $\underset{i \in I}{\cup} A_i$, $\underset{i \in I}{\oplus} \widehat{c_i}$, where $\widehat{c_i}$ is the cumulant multiplicative function induced by φ_i, for all $i \in I$. i.e*

$$c^{(n)}(a_1 \otimes ... \otimes a_n) = \begin{cases} c_i^{(n)}(a_1 \otimes ... \otimes a_n) & \text{if } \forall a_j \in A_i \\ 0_B & \text{otherwise.} \end{cases}$$

Now, we will observe the freeness over B ;

Definition 1.2. *Let (A, φ)be a NCPSpace over B.*

(1) Subalgebras containing B, $A_i \subset A$ $(i \in I)$ are free (over B) if we let $\varphi_i = \varphi \mid_{A_i}$, for all $i \in I$, then $*_i \varphi_i$ has its cumulant multiplicative function $\widehat{c}$ such that its restriction to $\underset{i \in I}{\cup} A_i$ is $\underset{i \in I}{\oplus} \widehat{c_i}$, where $\widehat{c_i}$ is the cumulant multiplicative function induced by each φ_i, for all $i \in I$.

(2) Sebsets X_i $(i \in I)$ are free (over B) if subalgebras A_i's generated by B and X_i's are free in the sense of (1). i.e If we let $A_i = A\lg(X_i, B)$, for all $i \in I$, then A_i's are free over B.

In [19], Speicher showed that the above combinatorial freeness with amalgamation can be used alternatively with respect to Voiculescu's original freeness with amalgamation.

Let (A, φ) be a NCPSpace over B and let $x_1, ..., x_s$ be B-valued random variables $(s \in \mathbb{N})$. Define $(i_1, ..., i_n)$-th moment of $x_1, ..., x_s$ by

$$\varphi(x_{i_1} b_{i_2} x_{i_2} ... b_{i_n} x_{i_n}),$$

for arbitrary $b_{i_2}, ..., b_{i_n} \in B$, where $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $\forall n \in \mathbb{N}$. Similarly, define a symmetric $(i_1, ..., i_n)$-th moment by the fixed $b_0 \in B$ by

$$\varphi(x_{i_1} b_0 x_{i_2} ... b_0 x_{i_n}).$$

If $b_0 = 1_B$, then we call this symmetric moments, trivial moments.

Cumulants defined below are main tool of combinatorial free probability theory ; in [20], we defined the $(i_1, ..., i_n)$-th cumulant of $x_1, ..., x_s$ by

$$k_n(x_{i_1}, ..., x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes ... \otimes b_{i_n} x_{i_n}),$$

for $b_{i_2}, ..., b_{i_n} \in B$, arbitrary, and $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $\forall n \in \mathbb{N}$, where $\widehat{c} = (c^{(n)})_{n=1}^{\infty}$ is the cumulant multiplicative function induced by φ. Notice that, by Möbius inversion, we can always take such B-value whenever we have $(i_1, ..., i_n)$-th moment of $x_1, ..., x_s$. And, vice versa, if we have cumulants, then we can always take moments. Hence we can define a symmetric $(i_1, ..., i_n)$-th cumulant by $b_0 \in B$ of $x_1, ..., x_s$ by

$$k_n^{symm(b_0)}(x_{i_1}, ..., x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_0 x_{i_2} \otimes ... \otimes b_0 x_{i_n}).$$

If $b_0 = 1_B$, then it is said to be trivial cumulants of $x_1, ..., x_s$.

By Speicher, it is shown that subalgebras A_i $(i \in I)$ are free over B if and only if all mixed cumulants vanish.

Proposition 1.1. *(See [19] and [20]) Let (A, φ) be a NCPSpace over B and let $x_1, ..., x_s \in (A, \varphi)$ be B-valued random variables ($s \in \mathbb{N}$). Then $x_1, ..., x_s$ are free if and only if all their mixed cumulants vanish.* $\square$

1.2. **Amalgamated R-transform Theory.**

In this section, we will define an R-transform of several B-valued random variables. Note that to study R-transforms is to study operator-valued distributions. R-transforms with single variable is defined by Voiculescu (over B, in particular, $B = \mathbb{C}$. See [20] and [11]). Over $\mathbb{C}$, Nica defined multi-variable R-transforms in [1]. In [20], we extended his concepts, over B. R-transforms of B-valued random variables can be defined as B-formal series with its $(i_1, ..., i_n)$-th coefficients, $(i_1, ..., i_n)$-th cumulants of B-valued random variables, where $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $\forall n \in \mathbb{N}$.

Definition 1.3. *Let (A, φ) be a NCPSpace over B and let $x_1, ..., x_s \in (A, \varphi)$ be B-valued random variables ($s \in \mathbb{N}$). Let $z_1, ..., z_s$ be noncommutative indeterminants. Define a moment series of $x_1, ..., x_s$, as a B-formal series, by*

$$M_{x_1,...,x_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,..,i_n \in \{1,...,s\}} \varphi(x_{i_1} b_{i_2} x_{i_2} ... b_{i_n} x_{i_n})\, z_{i_1} ... z_{i_n},$$

where $b_{i_2}, ..., b_{i_n} \in B$ are arbitrary for all $(i_2, ..., i_n) \in \{1, ..., s\}^{n-1}$, $\forall n \in \mathbb{N}$.

Define an R-transform of $x_1, ..., x_s$, as a B-formal series, by

$$R_{x_1,...,x_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} k_n(x_{i_1}, ..., x_{i_n})\, z_{i_1} ... z_{i_n},$$

with

$$k_n(x_{i_1}, ..., x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes ... \otimes b_{i_n} x_{i_n}),$$

where $b_{i_2}, ..., b_{i_n} \in B$ are arbitrary for all $(i_2, ..., i_n) \in \{1, ..., s\}^{n-1}$, $\forall n \in \mathbb{N}$. Here, $\widehat{c} = (c^{(n)})_{n=1}^{\infty}$ is a cumulant multiplicative function induced by φ in $I(A, B)$.

Denote a set of all B-formal series with s-noncommutative indeterminants ($s \in \mathbb{N}$), by Θ_B^s. i.e if $g \in \Theta_B^s$, then

$$g(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} b_{i_1,...,i_n}\, z_{i_1} ... z_{i_n},$$

where $b_{i_1,...,i_n} \in B$, for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $\forall n \in \mathbb{N}$. Trivially, by definition, $M_{x_1,...,x_s}, R_{x_1,...,x_s} \in \Theta_B^s$. By $\mathcal{R}_B^s$, we denote a set of all R-transforms of s-B-valued random variables. Recall that, set-theoratically,

$$\Theta_B^s = \mathcal{R}_B^s, \text{ sor all } s \in \mathbb{N}.$$

By the previous section, we can also define symmetric moment series and symmetric R-transform by $b_0 \in B$, by

$$M_{x_1,...,x_s}^{symm(b_0)}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} \varphi(x_{i_1} b_0 x_{i_2} ... b_0 x_{i_n})\, z_{i_1} ... z_{i_n}$$

and

$$R_{x_1,...,x_s}^{symm(b_0)}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,..,i_n \in \{1,...,s\}} k_n^{symm(b_0)}(x_{i_1}, ..., x_{i_n})\, z_{i_1} ... z_{i_n},$$

with

$$k_n^{symm(b_0)}(x_{i_1}, ..., x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_0 x_{i_2} \otimes ... \otimes b_0 x_{i_n}),$$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $\forall n \in \mathbb{N}$.

If $b_0 = 1_B$, then we have trivial moment series and trivial R-transform of $x_1, ..., x_s$ denoted by $M^t_{x_1,...,x_s}$ and $R^t_{x_1,...,x_s}$, respectively.

The followings are known in [19] and [20] ;

Proposition 1.2. *Let (A, φ) be a NCPSpace over B and let $x_1, ..., x_s, y_1, ..., y_p \in (A, \varphi)$ be B-valued random variables, where $s, p \in \mathbb{N}$. Suppose that $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_p\}$ are free in (A, φ). Then*

(1) $R_{x_1,...,x_s,y_1,...,y_p}(z_1, ..., z_{s+p}) = R_{x_1,...,x_s}(z_1, ..., z_s) + R_{y_1,...,y_p}(z_{s+1}, ..., z_{s+p})$.

(2) If $s = p$, then $R_{x_1+y_1,...,x_s+y_s}(z_1, ..., z_s) = (R_{x_1,...,x_s} + R_{y_1,...,y_s})(z_1, ..., z_s)$.
$\square$

The above proposition is proved by the characterization of freeness with respect to cumulants. i.e $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_p\}$ are free in (A, φ) if and only if their mixed cumulants vanish. Thus we have

$$
\begin{aligned}
&k_n(p_{i_1}, ..., p_{i_n}) = c^{(n)}(p_{i_1} \otimes b_{i_2} p_{i_2} \otimes ... \otimes b_{i_n} p_{i_n}) \\
&\quad = (\widehat{c_x} \oplus \widehat{c_y})^{(n)} (p_{i_1} \otimes b_{i_2} p_{i_2} \otimes ... \otimes b_{i_n} p_{i_n}) \\
&\quad = \begin{cases} k_n(x_{i_1}, ..., x_{i_n}) & \textit{or} \\ k_n(y_{i_1}, ..., y_{i_n}) \end{cases}
\end{aligned}
$$

and if $s = p$, then

$$
\begin{aligned}
&k_n(x_{i_1} + y_{i_1}, ..., x_{i_n} + y_{i_n}) \\
&= c^{(n)}\left((x_{i_1} + y_{i_1}) \otimes b_{i_2}(x_{i_2} + y_{i_2}) \otimes ... \otimes b_{i_n}(x_{i_n} + y_{i_n})\right) \\
&= c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes ... \otimes b_{i_n} x_{i_n}) + c^{(n)}(y_{i_1} \otimes b_{i_2} y_{i_2} \otimes ... \otimes b_{i_n} y_{i_n}) + [Mixed]
\end{aligned}
$$

where $[Mixed]$ is the sum of mixed cumulants, by the bimodule map property of $c^{(n)}$

$$= k_n(x_{i_1}, ..., x_{i_n}) + k_n(y_{i_1}, ..., y_{i_n}) + 0_B.$$

Now, we will define B-valued boxed convolution $\boxed{*}$, as a binary operation on Θ^s_B ; note that if $f, g \in \Theta^s_B$, then we can always choose free $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ in (some) NCPSpace over B, (A, φ), such that

$$f = R_{x_1,...,x_s} \quad \text{and} \quad g = R_{y_1,...,y_s}.$$

Definition 1.4. *(1) Let $s \in \mathbb{N}$. Let $(f, g) \in \Theta^s_B \times \Theta^s_B$. Define $\boxed{*} : \Theta^s_B \times \Theta^s_B \to \Theta^s_B$ by*

$$(f, g) = (R_{x_1,...,x_s}, R_{y_1,...,y_s}) \longmapsto R_{x_1,...,x_s} \boxed{*} R_{y_1,...,y_s}.$$

Here, $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are free in (A, φ). Suppose that

$coef_{i_1,..,i_n}\,(R_{x_1,...,x_s}) = c^{(n)}(x_{i_1} \otimes b_{i_2}x_{i_2} \otimes ... \otimes b_{i_n}x_{i_n})$
and
$coef_{i_1,...,i_n}(R_{y_1,...,y_s}) = c^{(n)}(y_{i_1} \otimes b'_{i_2}y_{i_2} \otimes ... \otimes b'_{i_n}y_{i_n}),$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$, *where* $b_{i_j}, b'_{i_n} \in B$ *arbitrary. Then*

$$coef_{i_1,...,i_n}\,(R_{x_1,...,x_s} \boxed{*}\, R_{y_1,...,y_s})$$

$$= \sum_{\pi \in NC(n)} (\widehat{c_x} \oplus \widehat{c_y})\,(\pi \cup Kr(\pi))(x_{i_1} \otimes y_{i_1} \otimes b_{i_2}x_{i_2} \otimes b'_{i_2}y_{i_2} \otimes ... \otimes b_{i_n}x_{i_n} \otimes b'_{i_n}y_{i_n})$$

$$\overset{denote}{=} \sum_{\pi \in NC(n)} \left(k_\pi \oplus k_{Kr(\pi)}\right)(x_{i_1}, y_{i_1}, ..., x_{i_n}y_{i_n}),$$

where $\widehat{c_x} \oplus \widehat{c_y} = \widehat{c}\mid_{A_x *_B A_y}$, $A_x = A\lg(\{x_i\}_{i=1}^s, B)$ *and* $A_y = A\lg(\{y_i\}_{i=1}^s, B)$ *and where* $\pi \cup Kr(\pi)$ *is an alternating union of partitions in* $NC(2n)$

Proposition 1.3. *(See [20])Let* (A, φ) *be a NCPSpace over* B *and let* $x_1, ..., x_s, y_1, ..., y_s \in (A, \varphi)$ *be* B*-valued random variables* ($s \in \mathbb{N}$)*. If* $\{x_1, ..., x_s\}$ *and* $\{y_1, ..., y_s\}$ *are free in* (A, φ)*, then we have*

$$k_n(x_{i_1}y_{i_1}, ..., x_{i_n}y_{i_n})$$

$$= \sum_{\pi \in NC(n)} (\widehat{c_x} \oplus \widehat{c_y})\,(\pi \cup Kr(\pi))(x_{i_1} \otimes y_{i_1} \otimes b_{i_2}x_{i_2} \otimes y_{i_2} \otimes ... \otimes b_{i_n}x_{i_n} \otimes y_{i_n})$$

$$\overset{denote}{=} \sum_{\pi \in NC(n)} \left(k_\pi \oplus k_{Kr(\pi)}^{symm(1_B)}\right)(x_{i_1}, y_{i_1}, ..., x_{i_n}, y_{i_n}),$$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $\forall n \in \mathbb{N}$, $b_{i_2}, ..., b_{i_n} \in B$, *arbitrary, where* $\widehat{c_x} \oplus \widehat{c_y} = \widehat{c}\mid_{A_x *_B A_y}$, $A_x = A\lg(\{x_i\}_{i=1}^s, B)$ *and* $A_y = A\lg(\{y_i\}_{i=1}^s, B)$. $\square$

This shows that ;

Corollary 1.4. *(See [20]) Under the same condition with the previous proposition,*

$R_{x_1,...,x_s} \boxed{*}\, R^t_{y_1,...,y_s} = R_{x_1y_1,...,x_sy_s}.$
$\square$

Notice that, in general, unless $b'_{i_2} = ... = b'_{i_n} = 1_B$ in B,

$$R_{x_1,...,x_s} \boxed{*}\, R_{y_1,...,y_s} \neq R_{x_1y_1,...,x_sy_s}.$$

However, as we can see above,

$$R_{x_1,...,x_s} \boxtimes R^t_{y_1,...,y_s} = R_{x_1y_1,...,x_sy_s}$$

and

$$R^t_{x_1,...,x_s} \boxtimes R^t_{y_1,...,y_s} = R^t_{x_1y_1,...,x_sy_s},$$

where $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are free over B. Over $B = \mathbb{C}$, the last equation is proved by Nica and Speicher in [1] and [17]. Actually, their R-transforms (over $\mathbb{C}$) is our trivial R-transforms (over $\mathbb{C}$).

2. Compatiblilty of a NCPSpace and an amalgamated NCPSpace over an algebra

In this Chapter, we will use notations defined in Section 1.3. In Section 2.1, we will introduce definitions about compatiblity. In Section 2.2, we will observe some cumulant-relations and in Section 2.3, based on Section 2.2, we will consider the R-transform calculus. In Section 2.4, we will observe some examples. Suppose that we have a unital algebra B and an algebra over B, A. Let $x_1, ..., x_s \in A$ be operators ($s \in \mathbb{N}$). Throught this paper, we will use the following notations ; for $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$,

$k_n(x_{i_1}, ..., x_{i_n})$: $(i_1, ..., i_n)$-th scalar-valued cumulants of $x_1, ..., x_s$,

in the sense of Speicher and Nica. (In our definition, they are $(i_1, ..., i_n)$-th $\mathbb{C}$-valued trivial cumulants of $x_1, ..., x_s$)

$K_n(x_{i_1}, ..., x_{i_n})$: $(i_1, ..., i_n)$-th B-valued cumulants of $x_1, ..., x_s$ and
$K^t_n(x_{i_1}, ..., x_{i_n})$: $(i_1, ..., i_n)$-th B-valued trivial cumulants of $x_1, ..., x_s$,

in the sense of Chapter 1.

$r_{x_1,...,x_s}(z_1, ..., z_s)$: the scalar-valued R-transform of $x_1, ..., x_s$,

in the sense of Speicher and Nica. (In our definition, $r_{x_1,...,x_s} = R^t_{x_1,...,x_s}$, over $\mathbb{C}$)

$R_{x_1,...,x_s}(z_1, ..., z_s)$: the B-valued R-transform of $x_1, ..., x_s$.

Of course, $r_{x_1,...,x_s}$ is an element in $\Theta^s_{\mathbb{C}}$ (i.e a formal series over $\mathbb{C}$) and $R_{x_1,...,x_s}$ is an element in Θ^s_B (i.e B-formal series). Simiralrly, we will denote a scalar-valued moment series of $x_1, ..., x_s$ and a B-valued moment series of $x_1, ..., x_s$ by

$$m_{x_1,...,x_s} \quad \text{and} \quad M_{x_1,...,x_s}$$

respectively.

2.1. **compatibility.**

In this section, we will introduce the compatibility. Let B be a unital algebra and let (A, E) be a NCPSpace over B. Also, let (A, φ) be a NCPSpace, in the sense of [1] and [17]. i.e an algebraic pair (A, φ) is a pairing of a unital algebra A and a linear functional $\varphi : A \rightarrow \mathbb{C}$. Notice that φ is nothing but a $\mathbb{C}$-functional, in the sense of Section 1.1. In [17], Śniady and Speicher introduced compatibility of (A, φ) and (A, E).

Definition 2.1. *Let B be a unital algebra contained in an algebra A, such that $1_B = 1_A$, and let (A, φ) be a NCPSpace. Let (A, E) is a NCPSpace over B. We say (A, φ) and (A, E) are compatible if*

$$\varphi(a) = \varphi\left(E(a)\right), \text{ for all } a \in A.$$

Suppose that there exists a linear functional $\psi : B \rightarrow \mathbb{C}$ such that

$$\varphi = \psi \circ E,$$

then, trivially, (A, φ) and (A, E) are compatible. However, in general, compatibility does not mean the existence of such ψ.

Example 2.1. *Let $F_k = < g_1, ..., g_k >$ be a free group with k-generators, $g_1, ..., g_k$. Assume that $< a, b >= F_2$. Then we can construct a unital group algebra $\mathbb{C}[F_2]$ and we can define a linear functional $tr : \mathbb{C}[F_2] \rightarrow \mathbb{C}$, by*

$$tr\left(\sum_{g \in F_2} \alpha_g g\right) = \begin{cases} \alpha_e & e \in F_2 \text{ is the identity} \\ & \\ 0 & \text{otherwise,} \end{cases}$$

for all $\sum_{g \in F_2} \alpha_g g \in \mathbb{C}[F_2]$. Now, we will put

$$G = < h \equiv aba^{-1}b^{-1} > \cong F_1 = \mathbb{Z}.$$

Then, similarly, we can define a unital algebra, $\mathbb{C}[F_1] = \mathbb{C}[\mathbb{Z}]$, as a subalgebra of $\mathbb{C}[F_2]$. Define a $\mathbb{C}[F_1]$-functional, $E : \mathbb{C}[F_2] \rightarrow \mathbb{C}[F_1]$ by

$$E\left(\sum_{g \in F_2} \alpha_g g\right) = \sum_{h \in F_1} \alpha_h h.$$

Then as a NCPSpace over $\mathbb{C}[F_1]$, $(\mathbb{C}[F_2],\ E)$ is compatible with $(\mathbb{C}[F_2], tr)$. Indeed,

$$tr\left(E(\sum_{g\in F_2}\alpha_g g)\right) = tr\left(\sum_{h\in F_1}\alpha_h h\right) = \alpha_e = tr\left(\sum_{g\in F_2}\alpha_g g\right),$$

for all $\sum_{g\in F_2}\alpha_g g \in \mathbb{C}[F_2] \hookrightarrow L(F_2)$.

The following lemma shows the relation between scalar-valued cumulants and a B-functional and the relation between scalar-valued moments and a B-functional. In fact the follwoing lemma is just an expression gotten from the Möbius inversion. So, at the first glance, this lemma is not so important. However, under the compatibility, if we know B-moments or B-cumulants, then we can get scalar-valued moments and scalar-valued cumulants by using them via the following lemma. For example, if we have a NCPSpace $(A_1 *_B A_2, \varphi)$ and an amalgamated NCPSpace over B, $(A_1 *_B A_2, E_1 * E_2)$ which are compatible and if we want to compute scalar-valued moments (resp. scalar-valued cumulants) of operators, then we can compute B-moments (resp. B-cumulants), first and then we can get the scalar-valued moment series (resp. the R-transform), by using the following lemma. In this case, it is more easy to compute B-moments (B-cumulants) than to compute scalar-valued moments (scalar-valued cumulants), directly, because of the relation depending on B, in $A_1 *_B A_2$.

Lemma 2.1. *Let B be a unital algebra and let A be an algebra over B (i.e $1_B = 1_A \in B \subset A$). Let (A,φ) be a NCPSpace and (A,E), a NCPSPace over B, with a B-functional, $E : A \to B$. If (A,φ) and (A,E) are compatible and if $x_1, ..., x_s \in A$ are operators ($s \in \mathbb{N}$), then the $(i_1, ..., i_n)$-th scalar-valued cumulant of $x_1, ..., x_s$ ($(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$) has the following relation with a B-functional $E : A \to B$;*

$$k_n(x_{i_1}, ..., x_{i_n}) = \sum_{\pi\in NC(n)}\left(\prod_{(v_1,...,v_k)\in\pi}\varphi\left(E(x_{v_1}...x_{v_k})\right)\right)\mu(\pi, 1_n).$$

Also, under the same assumption, the $(i_1, ..., i_n)$-th moment of $x_1, ..., x_s$ has the following relation with a B-functional $E : A \to B$;

$$\varphi(x_{i_1}...x_{i_n}) = \sum_{\pi\in NC(n)}\prod_{(v_1,...,v_k)\in\pi}\left(\sum_{\theta\in NC(k)}\prod_{(w_1,...,w_l)\in\theta}\varphi\left(E(x_{w_1}...x_{w_l})\right)\mu(\theta, 1_k)\right).$$

Proof. By the Möbius inversion, we have that

$$k_n(x_{i_1}, ..., x_{i_n}) = \sum_{\pi\in NC(n)}\varphi_\pi(x_{i_1}...x_{i_n})\mu(\pi, 1_n)$$

where $\varphi_\pi(...)$ is a partition-dependent scalar-valued moment of $x_1, ..., x_s$, in the sense of Speicher and Nica

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_V(x_{i_1}, ..., x_{i_n}) \right) \mu(\pi, 1_n)$$

where
$\varphi_V(x_{i_1}, ..., x_{i_n}) \overset{def}{=} \varphi(x_{v_1} ... x_{v_k})$,
if $V = (v_1, ..., v_k)$ is a block of π

$$= \sum_{\pi \in NC(n)} \left(\prod_{(v_1,...,v_k) \in \pi} \varphi(x_{v_1} ... x_{v_k}) \right) \mu(\pi, 1_n)$$
$$= \sum_{\pi \in NC(n)} \left(\prod_{(v_1,...,v_k) \in \pi} \varphi\left(E(x_{v_1} ... x_{v_k})\right) \right) \mu(\pi, 1_n).$$

Also, by the Möbius inversion, we can get that

$$\varphi(x_{i_1} ... x_{i_n}) = \sum_{\pi \in NC(n)} k_\pi(x_{i_1}, ..., x_{i_n})$$

where $k_\pi(...)$ is a partition-dependent scalar-valued cumulant of $x_1, ..., x_s$, in the sense of Speicher and Nica

$$= \sum_{\pi \in NC(n)} \prod_{V \in \pi} k_V(x_{i_1}, ..., x_{i_n})$$

where
$k_V(x_{i_1}, ..., x_{i_n}) \overset{def}{=} k_k(x_{v_1}, ..., x_{v_k})$,
if $V = (v_1, ..., v_k)$ is a block of π

$$= \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} k_k(x_{v_1}, ..., x_{v_k})$$
$$= \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \left(\sum_{\theta \in NC(k)} \prod_{(w_1,...,w_l) \in \theta} \varphi\left(E(x_{w_1} ... x_{w_l})\right) \mu(\theta, 1_k) \right),$$

by the previous discussion for scalar-valued cumulants. ∎

Example 2.2. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. Let $x_1, ..., x_7 \in A$.*

(1) We can compute scalar-valued $(1, 3, 4)$-cumulant of them as follows ;

$$k_3(x_1, x_3, x_4) = \sum_{\pi \in NC(3)} \left(\prod_{(v_1,...,v_k) \in \pi} \varphi\left(E(x_{v_1}, ..., x_{v_k})\right) \mu(\pi, 1_n) \right) \mu(\pi, 1_3)$$

$$= \varphi\left(E(x_1 x_3 x_4)\right) - \varphi\left(E(x_1)\right) \varphi\left(E(x_3 x_4)\right) - \varphi\left(E(x_1 x_3)\right) \varphi\left(E(x_4)\right)$$

$$-\varphi\left(E(x_1x_4)\right)\varphi\left(E(x_3)\right)+\varphi\left(E(x_1)\right)\varphi\left(E(x_3)\right)\varphi\left(E(x_4)\right).$$

(2) Now, suppose that each x_j, $j=1,...,7$, is centered over B. (i.e $E(x_j)=0_B$, for all $j=1,...,7$.) Then

$$\begin{aligned}
k_3(x_1,x_3,x_4) &= \varphi\left(E(x_1x_3x_4)\right)\\
&= \varphi\left(\sum_{\theta\in NC(3)}\widehat{C}(\theta)(x_1\otimes x_3\otimes x_4)\right)\\
&= \varphi\left(K_3(x_1,x_3,x_4)\right)+K_1(x_1)K_2(x_3,x_4)+K_2\left(x_1,K_1(x_3)x_4\right)\\
&\qquad +K_2(x_1,x_3)K_1(x_4)+K_1(x_1)K_1(x_2)K_1(x_3)\\
&= \varphi\left(K_3(x_1,x_3,x_4)\right),
\end{aligned}$$

since $E(x_j)=K_1(x_j)$, for all $j=1,...,7$. So, if x_j's are centered, then

$$k_3(x_1,x_3,x_4)=\varphi\left(K_3(x_1,x_3,x_4)\right).$$

We hope that scalar-valued cumulants and operator-valued cumulants have nice property under the compatibility such as

$$\varphi\left(K_n^t(x_{i_1},...,x_{i_n})\right)=k_n(x_{i_1},...,x_{i_n}).$$

However, in general, the above relation does NOT hold true (See the above example (1)). Observe the following ; if $x,y\in A$ are operators, then

$$\begin{aligned}
\varphi\left(K_2^t(x,y)\right) &= \varphi\left(E(xy)-E(x)E(y)\right)\\
&= \varphi\left(E(xy)\right)-\varphi\left(E(x)E(y)\right)
\end{aligned}$$

and

$$\begin{aligned}
k_2(xy) &= \varphi(xy)-\varphi(x)\varphi(y)\\
&= \varphi\left(E(xy)\right)-\varphi\left(E(x)\right)\varphi\left(E(y)\right).
\end{aligned}$$

Thus to get $\varphi\left(K_2^t(x,y)\right)=k_2(x,y)$, we need the following eaulity ;

$$\varphi\left(E(x)E(y)\right)=\varphi\left(E(x)\right)\varphi\left(E(y)\right),$$

since

$$\begin{aligned}
&\varphi\left(K_2^t(x,y)\right)-k_2(x,y)\\
&\quad=\varphi\left(E(xy)\right)-\varphi\left(E(x)E(y)\right)-\left(\varphi\left(E(xy)\right)-\varphi\left(E(x)\right)\varphi\left(E(y)\right)\right)\\
&\quad=\varphi\left(E(x)\right)\varphi\left(E(y)\right)-\varphi\left(E(x)E(y)\right).
\end{aligned}$$

If $\varphi:A\to\mathbb{C}$ is a homomorphism, then it happens, however, in general, we cannot guarantee the above equality. For example, suppose that we have a UHF-algebra A and $B=M_N(\mathbb{C})$ $(N\neq 1)$ and assume that $E_N:A\to M_N(\mathbb{C})$ is a

canonical conditional expectation and $\varphi = \lim_{k \to \infty} \varphi_k : A \to \mathbb{C}$ is a normalized trace. Let

$$E_N(x) = \begin{pmatrix} 0 & & & & O \\ 1 & 0 & & & \\ 0 & \ddots & 0 & & \\ & \ddots & \ddots & \ddots & \\ O & & 0 & 1 & 0 \end{pmatrix} \text{ and } E_N(y) = \begin{pmatrix} 0 & 1 & 0 & & O \\ & 0 & \ddots & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & 1 \\ O & & & & 0 \end{pmatrix}$$

in $M_N(\mathbb{C})$. Then

$$\varphi\left(E(x)E(y)\right) = \varphi_N\left(\begin{pmatrix} 0 & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{pmatrix}\right) = \frac{N-1}{N}$$

and

$$\varphi\left(E(x)\right)\varphi\left(E(y)\right) = \varphi_N\left(E(x)\right) \cdot \varphi_N\left(E(y)\right) = 0.$$

More generally, to satisfy

$$\varphi\left(K_n^t(x_{i_1}, ..., x_{i_n})\right) = k_n(x_{i_1}, ..., x_{i_n}),$$

operators $x_1, ..., x_s \in A$ should satisfy ;

$$(*) : \varphi\left(\prod_{V \in \pi(o)} \widehat{E}(\pi\mid_V)(x_{i_1} \otimes ... \otimes \widehat{E}(\pi\mid_W)(x_{i_1} \otimes ... \otimes x_{i_n})x_{i_j} \otimes ... \otimes x_{i_n})\right)$$
$$= \prod_{V \in \pi(o)} \varphi\left(\widehat{E}(\pi\mid_V)(x_{i_1} \otimes ... \otimes \varphi\left(\widehat{E}(\pi\mid_W)(x_{i_1} \otimes ... \otimes x_{i_n})\right) x_{i_j} \otimes ... \otimes x_{i_n})\right),$$

for any $\pi \in NC(n)$, where W is an inner block having its outer block V, for the fixed $\pi \in NC(n)$.

Proposition 2.2. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. Assume that operators $x_1, ..., x_s \in A$ $(s \in \mathbb{N})$ satisfy*

$$(*) : \varphi\left(\prod_{V \in \pi(o)} \widehat{E}(\pi\mid_V)(x_{i_1} \otimes ... \otimes \widehat{E}(\pi\mid_W)(x_{i_1} \otimes ... \otimes x_{i_n})x_{i_j} \otimes ... \otimes x_{i_n})\right)$$
$$= \prod_{V \in \pi(o)} \varphi\left(\widehat{E}(\pi\mid_V)(x_{i_1} \otimes ... \otimes \varphi\left(\widehat{E}(\pi\mid_W)(x_{i_1} \otimes ... \otimes x_{i_n})\right) x_{i_j} \otimes ... \otimes x_{i_n})\right),$$

for any $\pi \in NC(n)$, for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$, where $W \in \pi(i)$ is an inner block of $V \in \pi(o)$, for the fixed $\pi \in NC(n)$. Then

$\varphi\left(K_n^t(x_{i_1}, ..., x_{i_n})\right) = k_n(x_{i_1}, ..., x_{i_n}),$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$. *In particular,*

$r_{x_1,...,x_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} \varphi\left(coef_{i_1,...,i_n}(R^t_{x_1,...,x_s})\right) z_{i_1}...z_{i_n}.$

□

$\boxed{\textbf{Notation}}$ We will say that operators $x_1, ..., x_s \in A$ satisfy property (*) if $x_1, ..., x_s$ satisfy the relation (*) introduced in the previous proposition.

Corollary 2.3. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. If a linear functional $\varphi : A \to \mathbb{C}$ is an algebra homomorphism, then, for $x_1, ..., x_s \in A$ ($s \in \mathbb{N}$),*

$r_{x_1,...,x_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} \varphi\left(coef_{i_1,...,i_n}(R^t_{x_1,...,x_s})\right) z_{i_1}...z_{i_n}.$

□

This shows that it is difficult to verify the relation between $r_{x_1,...,x_s}$ and $R_{x_1,...,x_s}$. This also says that B-freeness and scalar-valued freeness have a deep gap, in general.

Theorem 2.4. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. If $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ are B-free families of B-valued random variables ($s, t \in \mathbb{N}$) and if $x_1, ..., x_s, y_1, ..., y_t$ satisfy property(*), then $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ are free in (A, φ). In particular, if a linear functional $\varphi : A \to \mathbb{C}$ is nondegenerated and if $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ are central in the sense of [20], then the converse also holds true.*

Proof. Assume that operators $x_1, ..., x_s, y_1, ..., y_t$ satisfies property (*). Then, by the previous proposition, we have that

$k_n(p_1, ..., p_n) = \varphi\left(K_n^t(p_1, ..., p_n)\right),$

where $p_1, ..., p_n \in \{x_1, ..., x_s\} \cup \{y_1, ..., y_t\}$. Thus, if $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ are B-free over B, in (A, E), then a mixed cumulant of $x_1, ..., x_s, y_1, ..., y_t$ vanishs ;

$k_n(p_1, ..., p_n) = \varphi(0_B) = 0.$

This shows that all mixed scalar-valued cumulants of $x_1, ..., x_s, y_1, ..., y_t$ vanish, too. Equivalently, $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ are free in (A, φ).

Now, suppose that the linear functional φ is nondegenerated and assume that $x_1, ..., x_s$ and $y_1, ..., y_t$ are central. By the property "central", B-freeness is equivalent to the statement [all mixed trivial cumulants vanish] (See [20] and [29]). So, we have that ; for the mixed case,

$$0 = k_n(p_1, ..., p_n) = \varphi\left(K_n^t(p_1, ..., p_n)\right)$$
$$\Longleftrightarrow$$
$$K_n^t(p_1, ..., p_n) = 0_B,$$

by the nondegeneratedness of φ. Therefore, $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ are B-free over B in (A, E) ∎

Also, the above theorem says that even under the compatibility, B-freeness and scalar-valued freeness have a deep gap, in general. Actually, the property (*) is a very powerful assumption. We redefine the property (*) which is the extended concept of having the property (*)

Definition 2.2. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. We say that operators $x_1, ..., x_s \in A$ satisfy the property (*), if*

$$k_n(x_{i_1}, ..., x_{i_n}) = \varphi\left(K_n^t(x_{i_1}, ..., x_{i_n})\right),$$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$.

2.2. **R-transformTheory under the compitablity : Connection between scalar-valued R-transforms and operator-valued Moment Series.**

In this section, we will observe the connections between scalar-valued R-transforms and operator-valued moment series under the compatibility of a NCPSpace and amalgamated NCPSpace. This connection will be used for studying scalar-valued distributions of (some) operators and operator-valued distribution of those operators. From the previous section, we can compute operator-valued R-transforms and scalar-valued R-transforms. Let B be a unital algebra.

Theorem 2.5. *(See Theorem 6, in [17]) Let B be a unital algebra and A, an algebra over B. Let (A, φ) and (A, E) be compatible and let $x_1, ..., x_s \in A$ be operators ($s \in \mathbb{N}$). Then $\{x_1, ..., x_s\}$ and B are free in (A, φ) if and only if we have that*

$$K_n(x_{i_1}, ..., x_{i_n}) = \left(\varphi(b_{i_2}) \cdots \varphi(b_{i_n})\right) k_n(x_{i_1}, ..., x_{i_n}) \ \cdot 1_B \in B,$$

where

$$K_n(x_{i_1},...,x_{i_n}) = C^{(n)}\left(x_{i_1} \otimes b_{i_2}x_{i_2} \otimes ... \otimes b_{i_n}x_{i_n}\right),$$

for arbitrary $b_{i_2},...,b_{i_n} \in B$, *and for all* $(i_1,...,i_n) \in \{1,...,s\}^n$, $n \in \mathbb{N}$. $\square$

The above theorem shows that if some family of operators in A and a unital algebra B, in A, are free in (A,φ), then an operator-valued (B-valued) cumulants of those operators are easily expressed in termes of multiplication of some scalar-value and scalar-valued trivial cumulants of those operators.

Corollary 2.6. *Let* B *be a unital algebra and* A, *an algebra over* B. *Let* (A,φ) *and* (A,E) *be compatible NCPSpace and NCPSpace over* B. *If* $x_1,...,x_s \in A$ *are operators* ($s \in \mathbb{N}$) *and if* $\{x_1,...,x_s\}$ *and* B *are free in* (A,φ), *then there exist* $\alpha_{i_1,...,i_n} \in \mathbb{C}$, *for all* $(i_1,...,i_n) \in \{1,...,s\}^n$, $n \in \mathbb{N}$, *such that*

$$R_{x_1,...,x_s}(z_1,...,z_s) = \textstyle\sum_{n=1}^{\infty} \displaystyle\sum_{i_1,...,i_n \in \{1,...,s\}} \left(\alpha_{i_1,...,i_n}\, k_n(x_{i_1},...,x_{i_n}) \cdot 1_B\right) z_{i_1}...z_{i_n},$$

in Θ_B^s, *as a* B*-formal series.* $\square$

By the previous theorem, in particular, we can characterize $\alpha_{i_1,...,i_n} \in \mathbb{C}$, for each $(i_1,...,i_n) \in \{1,...,s\}^n$; $n \in \mathbb{N}$, as follows ;

$$\alpha_{i_1,...,i_n} = \varphi(1_B) \cdot \varphi(b_{i_2}) \cdots \varphi(b_{i_n}) \in \mathbb{C},$$

where $b_{i_2},...,b_{i_n} \in B \subset A$ are determined in

$$K_n(x_{i_1},...,x_{i_n}) = C^{(n)}\left(x_{i_1} \otimes b_{i_2}x_{i_2} \otimes ... \otimes b_{i_n}x_{i_n}\right).$$

Notice that the above B-valued R-transform of $x_1,...,x_s$, $R_{x_1,...,x_s}$ is a B-formal series, but each coefficient has the form of $\alpha \cdot 1_B$, where $\alpha \in \mathbb{C}$. This says that we can regard the R-transform, $R_{x_1,...,x_s}$, as a scalar-valued formal series, with its $(i_1,...,i_n)$-th coefficients

$$k_n(x_{i_1},\, \varphi(b_{i_2})x_{i_2},...,\varphi(b_{i_n})x_{i_n}) = \left(\varphi(b_{i_2})...\varphi(b_{i_n})\right) k_n(x_{i_1},...,x_{i_n}).$$

Now, we will denote boxed convolution and amalgamated boxed convolution over an algebra B by $\boxed{*}$ and $\boxed{*}_B$, respectively.

Theorem 2.7. *Let* B *be a unital algebra and* A, *an algebra over* B. *Let* (A,φ) *and* (A,E) *be compatible. If* $x_1,...,x_s \in A$ *are operators* ($s \in \mathbb{N}$) *and if* $\{x_1,...,x_s\}$ *are free from* B, *in* (A,φ), *then there exists a* B*-formal series* $g \in \Theta_B^s$ *such that*

$$r_{x_1,...,x_s}(z_1,...,z_s) = \textstyle\sum_{n=1}^{\infty} \displaystyle\sum_{i_1,...,i_n \in \{1,...,s\}} [\text{scalar part of } K_n^t(x_{i_1},...,x_{i_n})] z_{i_1}...z_{i_n}.$$

Proof. By the previous theorem, we have that, for $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$,

$$K_n^t(x_{i_1}, ..., x_{i_n}) = k_n(x_{i_1}, ..., x_{i_n}) \cdot 1_B \equiv c_{i_1,...,i_n} \cdot 1_B.$$

Therefore, in this case, we can get that

$$\begin{aligned} r_{x_1,...,x_s}(z_1, ..., z_s) &= \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} k_n(x_{i_1}, ..., x_{i_n}) z_{i_1}...z_{i_n} \\ &= \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} c_{i_1,...,i_n}\, z_{i_1}...z_{i_n} \\ &= \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} [\text{scalar part of } K_n^t(x_{i_1}, ..., x_{i_n})] z_{i_1}...z_{i_n} \end{aligned}$$ ∎

Remark 2.1. *Remark that if a set of operators $\{x_1, ..., x_s\} \subset A$ is free from B, in (A, φ), then $x_1, ..., x_s$ satisfy the property (*). Indeed,*

$$\begin{aligned} \varphi\left(K_n^t(x_{i_1}, ..., x_{i_n})\right) &= \varphi\left(k_n(x_{i_1}, ..., x_{i_n}) \cdot 1_B\right) \\ &= k_n(x_{i_1}, ..., x_{i_n}) \cdot \varphi(1_B) \\ &= k_n(x_{i_1}, ..., x_{i_n}), \end{aligned}$$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$.

Corollary 2.8. *Let B be a unital algebra and A, an algebra over B. Suppose a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. Let $x_1, ..., x_s, y_1, ..., y_s \in A$ be operators such that $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are B-free in (A, E). If $\{x_1, ..., x_s, y_1, ..., y_s\}$ and B are free in (A, φ), then*

(1) $r_{x_1,...,x_s,y_1,...,y_s}(z_1, ..., z_{2s}) = r_{x_1,..,x_s}(z_1, ..., z_s) + r_{y_1,...,y_s}(z_{s+1}, ..., z_{2s})$.

(2) $r_{x_1+y_1,...,x_s+y_s}(z_1, ..., z_s) = r_{x_1,...,x_s}(z_1, ..., z_s) + r_{y_1,...,y_s}(z_1, ..., z_s)$

(3) $r_{x_1y_1,...,x_sy_s}(z_1, ..., z_s) = \left(r_{x_1,...,x_s} \boxed{*}\, r_{y_1,...,y_s}\right)(z_1, ..., z_s)$.

Proof. (1) Since a set of operators in A, $\{x_1, ..., x_s, y_1, ..., y_s\}$ are free from B, in (A, φ), by the previous theorem, we can get that

$$r_{x_1,...,x_s,y_1,...,y_s}(z_1, ..., z_{2s}) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,2s\}} c_{i_1,...,i_n}\, z_{i_1}...z_{i_n}$$

where
$c_{i_1,...,i_n}$ = the scalar part of $K_n^t(p_{i_1}, ..., p_{i_n})$
with

$$K_n^t(p_{i_1}, ..., p_{i_n}) = \begin{cases} K_n^t(x_{i_1}, ..., x_{i_n}) & \text{or} \\ \\ K_n^t(y_{i_1}, ..., y_{i_n}), \end{cases}$$

by the B-freeness of $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$. Put

$$K_n^t(x_{i_1}, ..., x_{i_n}) = c^x_{i_1,...,i_n} \cdot 1_B \quad \text{and} \quad K_n^t(y_{i_1}, ..., y_{i_n}) = c^y_{i_1,...,i_n} \cdot 1_B,$$

for all $(i_1, ..., i_n) \in \{1, ..., 2s\}^n$, $n \in \mathbb{N}$.

Thus, we have that

$$\begin{aligned} &r_{x_1,...,x_s,y_1,...,y_s}(z_1, ..., z_{2s}) \\ &\qquad = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,2s\}} \left(c^x_{i_1,...,i_n} + c^y_{i_1,...,i_n}\right) z_{i_1}...z_{i_n} \\ &\qquad = r_{x_1,...,x_s}(z_1, ..., z_s) + r_{y_1,...,y_s}(z_{s+1}, ..., z_{2s}). \end{aligned}$$

(2) Since $\{x_1, ..., x_s, y_1, ..., y_s\}$ are free from B, in (A, φ), so are $\{x_1 + y_1, ..., x_s + y_s\}$ and B. Thus

$$r_{x_1+y_1,...,x_s+y_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} c_{i_1,...,i_n} z_{i_1}...z_{i_n}$$

where
$c_{i_1,...,i_n}$ = the scalar part of $K_n^t(x_{i_1} + y_{i_1}, ..., x_{i_n} + y_{i_n})$.

By the B-freeness of $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$,

$$K_n^t(x_{i_1} + y_{i_1}, ..., x_{i_n} + y_{i_n}) = K_n^t(x_{i_1}, ..., x_{i_n}) + K_n^t(y_{i_1}, ..., y_{i_n}),$$

for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $n \in \mathbb{N}$. Notice that both $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are free from B, in (A, φ). Hence

$$K_n^t(x_{i_1}, ..., x_{i_n}) = k_n(x_{i_1}, ..., x_{i_n}) \cdot 1_B \equiv c^x_{i_1,...,i_n} \cdot 1_B$$
and
$$K_n^t(y_{i_1}, ..., y_{i_n}) = k_n(y_{i_1}, ..., y_{i_n}) \cdot 1_B \equiv c^y_{i_1,...,i_n} \cdot 1_B.$$

Therefore,

$$c_{i_1,...,i_n} \cdot 1_B = \left(c^x_{i_1,...,i_n} + c^y_{i_1,...,i_n}\right) \cdot 1_B$$
and
$$r_{x_1+y_1,...,x_s+y_s}(z_1, ..., z_s) = \left(r_{x_1,...,x_s} + r_{y_1,...,y_s}\right)(z_1, ..., z_s).$$

(3) Similarly, $\{x_1y_1, ..., x_sy_s\}$ and B are free in (A, φ). Thus

$$r_{x_1y_1,...,x_sy_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1,...,i_n \in \{1,...,s\}} c_{i_1,...,i_n} z_{i_1}...z_{i_n}$$
with
$c_{i_1,...,i_n}$ = the scalar part of $K_n^t(x_{i_1}y_{i_1}, ..., x_{i_n}y_{i_n})$.

By the B-freeness of $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$, we have that

$$K_n^t(x_{i_1}y_{i_1}, ..., x_{i_n}y_{i_n}) = C^{(n)}\left(x_{i_1}y_{i_1} \otimes ... \otimes x_{i_n}y_{i_n}\right)$$

$$= \sum_{\pi \in NC(n)} \left(\widehat{C_x} \oplus \widehat{C_y}\right)(\pi \cup Kr(\pi))(x_{i_1} \otimes y_{i_1} \otimes ... \otimes x_{i_n} \otimes y_{i_n})$$

$$= coef_{i_1,...,i_n}\left(R^t_{x_1,...,x_s} \boxed{*}_B \; R^t_{y_1,...,y_s}\right),$$

where $\widehat{C_x} \oplus \widehat{C_y} = \widehat{C} \mid_{A_x *_B A_y}$, $A_x = A\lg\left(\{x_j\}_{j=1}^s, B\right)$ and $A_y = A\lg\left(\{y_j\}_{j=1}^s, B\right)$. Notice that, since both $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are free from B, in (A, φ),

$K_N^t\left(x_{j_1}, ..., x_{j_N}\right) = k_N\left(x_{j_1}, ..., x_{j_N}\right) \cdot 1_B$
and
$K_N^t\left(y_{j_1}, ..., y_{j_N}\right) = k_N\left(y_{j_1}, ..., y_{j_N}\right) \cdot 1_B.$

(i.e they are multiplications of scalar-value and 1_B and hence $x_1, ..., x_s$ and $y_1, ..., y_s$ are central !) Thus we can get that

$$\sum_{\pi \in NC(n)} \left(\widehat{C_x} \oplus \widehat{C_y}\right)(\pi \cup Kr(\pi))(x_{i_1} \otimes y_{i_1} \otimes ... \otimes x_{i_n} \otimes y_{i_n})$$

$$= \sum_{\pi \in NC(n)} \left(\widehat{C_x}(\pi)(x_{i_1} \otimes ... \otimes x_{i_n})\right)\left(\widehat{C_y}(Kr(\pi))(y_{i_1} \otimes ... \otimes y_{i_n})\right)$$
$$= \sum_{\pi \in NC(n)} \left(k_\pi(x_{i_1}, ..., x_{i_n}) \cdot 1_B\right)\left(k_{Kr(\pi)}(y_{i_1}, ..., y_{i_n}) \cdot 1_B\right).$$

Therefore,

$$r_{x_1y_1,...,x_sy_s}(z_1, ..., z_s) = \left(r_{x_1,...,x_s} \boxed{*} \, r_{y_1,...,y_s}\right)(z_1, ..., z_s) \;\blacksquare$$

From now, we will consider the one-variable scalar-valued R-transforms and operator-valued ones. As we have seen, if we have compatible (A, φ) and (A, E), over B and if $x_1, ..., x_s$ are operators $(s \in \mathbb{N})$ in A, then $(i_1, ..., i_n)$-th scalar-valued cumulants of $x_1, ..., x_s$ are

$$k_n(x_{i_1}, ..., x_{i_n}) = \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \varphi\left(E(x_{v_1}...x_{v_k})\right)\mu(\pi, 1_n).$$

Now, let $x \in A$ be an operator. Then we can consider the n-th cumulants of x, for all $n \in \mathbb{N}$ and, under the compatibility, we can get that

$$k_n\left(\underbrace{x, ..., x}_{n-times}\right) = \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \varphi\left(E(x^k)\right)\mu(\pi, 1_n).$$

Later, what we are most interested in is to compute n-th moments of an operator x in A. Since we have a method to compute m-th cumulants of x, as we described above, we can compute n-th moments, by using the Möbius inversion. Actually, we have that

$$\varphi(x^n) = \sum_{\pi \in NC(n)} k_\pi \left(\underbrace{x, ...,x}_{n-timse} \right) = \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} k_k \left(\underbrace{x, ...,x}_{n-times} \right).$$

So,

$$\varphi(x^n) = \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \left(\sum_{\theta \in NC(k)} \prod_{(j_1,...,j_l) \in \theta} \varphi\left(E(x^l)\right) \mu(\theta, 1_k) \right).$$

Lemma 2.9. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. If $x \in A$ is an operator, then we can compute n-th cumulants ($n \in \mathbb{N}$) of x by*

$$k_n \left(\underbrace{x, ...,x}_{n-times} \right) = \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \varphi\left(E(x^k)\right) \mu(\pi, 1_n).$$

Also, we can compute n-th moments of x by

$$\varphi(x^n) = \sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \left(\sum_{\theta \in NC(k)} \prod_{(j_1,...,j_l) \in \theta} \varphi\left(E(x^l)\right) \mu(\theta, 1_k) \right).$$

□

By the previous lemma, we can find a scalar-valued R-transform of x and a scalar-valued moment series of x, under the compatibility ;

Proposition 2.10. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. If $x \in A$ is an operator, then its scalar-valued R-transform and moment series as follows ;*

$$r_x(z) = \sum_{n=1}^{\infty} \left(\sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \varphi\left(coef_k(M_x)\right) \mu(\pi, 1_n) \right) z^n$$

and

$$m_x(z) = \sum_{n=1}^{\infty} \left(\sum_{\pi \in NC(n)} \prod_{(v_1,...,v_k) \in \pi} \left(\sum_{\theta \in NC(k)} \prod_{(j_1,...,j_l) \in \theta} \varphi\left(coef_l(M_x)\right) \mu(\theta, 1_k) \right) \right) z^n.$$

In particular, a scalar-valued moment series of x has the following relation ;

$$m_x(z) = r_x\left(z \cdot m_x(z)\right).$$

Proof. By the previous lemma, we have, for each $n \in \mathbb{N}$,

$$coef_n(r_x(z)) = k_n\left(\underbrace{x, ..., x}_{n-times}\right)$$
and
$$coef_n\left(m_x(z)\right) = \varphi(x^n).$$

In general, by Nica (See [1]), if $x_1, ..., x_s \in (A, \varphi)$ are scalar-valued random variables ($s \in \mathbb{N}$), then

$$m_{x_1,...,x_s}(z_1, ..., z_s) = r_{x_1,...,x_s}\left(z_1\left(m_{x_1,...,x_s}(z_1, ..., z_s)\right), ..., z_s\left(m_{x_1,...,x_s}(z_1, ..., z_s)\right)\right),$$

since $m_{x_1,...,x_s} = r_{x_1,...,x_s} \boxed{*}\ Zeta.$ ∎

2.3. **Compatible Subalgebras, in (A, φ), relative to (A, E).**

Until now, we observed the compatibility of a NCPSpace (A, φ) and an amalgamated NCPSpace (A, E). It will be interesting to find an "compatible part" in (A, φ), when we have arbitrary (A, φ) and (A, E). In this section, we will not assume that (A, φ) and (A, E) are compatible. They are just given NCPSpace and an amalgmated NCPSpace over an algebra. We want to observe a "compatible part" of (A, φ), with respect to (A, E).

Definition 2.3. *Let B be a unital algebra and A, an algebra over B and let (A, φ) be a NCPSpace and (A, E), a NCPSpace over B. A subalgebra $A^o \subset A$ is called a compatible Subalgebra of (A, φ), relative to (A, E) if*

$$\varphi(x) = \varphi\left(E(x)\right), \text{ for all } x \in A^o.$$

Remark 2.2. *Let B be a unital algebra and A, an algebra over B and let (A, φ) and (A, E) be given NCPSpace and NCPSpace over B, respectively. Then there always exists a compatible subalgebra relative to (A, E) called the scalar-valued compatible subalgebra which is isomorphic to $\mathbb{C}$,*

$$\mathbb{C} \cdot 1_B = \{\alpha \cdot 1_B : \alpha \in \mathbb{C}\} \subset B \subset A,$$

denoted by S. Clearly, for any $\alpha \cdot 1_B = \alpha \cdot 1_A \in S$ satisfies

$$\varphi\left(\alpha \cdot 1_B\right) = \alpha = \varphi\left(E(\alpha \cdot 1_B)\right).$$

Thus, there always exists compatible subalgebra relative to (A, E)*, in* (A, φ)*.*

The next proposition also shows that there always exists a compatible subalgebra of a NCPSpace (A, φ) relative to an amalgamated NCPSpace (A, E).

Proposition 2.11. *Let* B *be a unital algebra and* A*, an algebra over* B*. Let* (A, φ) *be a NCPSpace and let* (A, E) *be a NCPSpace over* B*, with its* B*-functional* $E : A \to B$*. Then* $B \subset A$ *is a compatible subalgebra of* A *relative to* (A, E)*.*

Proof. Clearly, we can get that

$$\varphi(b) = \varphi\left(E(b)\right), \text{ for all } b \in B,$$

since $E(b) = b$, for all $b \in B$. ∎

Definition 2.4. *Let* B *be a unital algebra and let* (A, E) *be a NCPSpace over* B*. Let* $x \in (A, E)$ *be a* B*-valued random variable. It is said that a* B*-valued random variable* x *has scalar-valued property if there exists* $\alpha_n \in \mathbb{C}$ *such that*

$$E(x^n) = \alpha_n \cdot 1_B \in B,$$

for all $n \in \mathbb{N}$*.*

Notice that we only use **trivial** moments of x, not general moments of x, in the above definition.

Theorem 2.12. *Let* B *be a unital algebra and* A*, an algebra over* B*. Let* (A, φ) *be a NCPSpace and* (A, E)*, a NCPSpace over* B*, with its* B*-functional* $E : A \to B$*. Suppose that* $x_0 \in A$ *is an operator which is* B*-central, as a* B*-valued random variable (i.e* $x_0 b = b x_0$*, for all* $b \in B$*) and assume that*

$$\varphi(x_0^n) = \varphi\left(E(x_0^n)\right), \textit{ for all } n \in \mathbb{N}.$$

If x_0 *has scalar-valued property determined by a sequence* $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$*, then a free product of an algebra generated by* $\{x_0\}$ *and* B*, denoted by* $A^o = A\lg\left(\{x_0\}\right) * B \subset A$*, is a compatible subalgebra relative to* (A, E)*.*

Proof. It suffices to show that

$$\varphi\left(b_1 {x_0}^{k_1} b_2 {x_0}^{k_2} ... b_n {x_0}^{k_n}\right) = \varphi\left(E(b_1 x_0^{k_1} b_2 x_0^{k_2} ... b_n x_0^{k_n})\right),$$

for all $n \in \mathbb{N}$, where $k_1, ..., k_n \in \mathbb{N}$. Remark that here φ can be regarded as $\varphi \mid_{A^o}$, where $A^o = A\lg\left(\{x_0\}\right) * B \subset A$. The above equation can be rewritten by

$$\varphi\left((b_1b_2...b_n)x_0^{k_1+...+k_n}\right) = \varphi\left(E(b_1...b_n \cdot x_0^{k_1+...+k_n})\right)$$
$$\Longleftrightarrow$$
$$\varphi\left(bx_0^N\right) = \varphi\left(E(bx_0^N)\right),$$

where $b = b_1...b_n \in B$ and $N = \sum_{j=1}^{n} k_j$, since x_0 is B-central. Fix $n \in \mathbb{N}$. By the above equation, it is enough to show that

$$\varphi\left(bx_0^N\right) = \varphi\left(bE(x_0^N)\right), \text{ for } b \in B \text{ arbitrary.}$$

By scalar-valued property determined by $(\alpha_n)_{n=1}^{\infty}$ and by the assumption that $\alpha_n \cdot 1_B = E(x_0^n)$, for all $n \in \mathbb{N}$,

$$\alpha_n = \varphi\left(E(x_0^n)\right), \text{ for all } n \in \mathbb{N}.$$
And
$$\varphi\left(bE(x_0^N)\right) = \alpha_N \cdot \varphi(b), \text{ for all } N \in \mathbb{N}.$$

Now, consider that, if we put $y = x_0^N \in A^o$, then

$$\begin{aligned}\varphi\left(bx_0^N\right) = \varphi(by) &= \sum_{\pi \in NC(2)} k_\pi(b, y) \\ &= k_2(b, y) + k_1(b)k_1(y) = k_2(b, y) + \varphi(b)\varphi(y) \\ &= 0 + \varphi(b)\varphi(y)\end{aligned}$$

since b and y are free in $(A^o, \varphi) \equiv \left(A\lg(\{x_0\}) * B,\ \varphi\mid_{A\lg(\{x_0\})*B}\right)$

$$\begin{aligned}&= \varphi(b)\varphi\left(x_0^N\right) = \varphi(b)\varphi\left(E(x_0^N)\right) \\ &= \varphi(b) \cdot \alpha_N,\end{aligned}$$
hence
$$\varphi\left(bx_0^N\right) = \alpha_N \cdot \varphi(b), \text{ for all } N \in \mathbb{N}.$$

Therefore, for any $n \in \mathbb{N}$,
$$\varphi\left(bx_0^n\right) = \varphi\left(bE(x_0^n)\right). \blacksquare$$

The above theorem shows how to construct a compatible subalgebra of A, relative to (A, E). But this is not a general method to get a compatible subalgebra of (A, φ), relative to (A, E). And this construction is very artificial. But this provides us one way to construct a compatible subalgebra containing B in (A, φ), relative to (A, E). We can see the following special case ;

Example 2.3. *Let B, A, (A, φ) and (A, E) be given as before. Let $x_0 \in A$ be a B-semicircular element in (A, E), in the sense that only nonvanishing B-cumulant is the second one, such that*

$$K_2^t(x_0, x_0) = t \cdot 1_B, \text{ for some } t \in \mathbb{C}.$$
and
$$x_0 b = b x_0, \text{ for all } b \in B.$$

Then x_0 satisfies the scalar-valued property. i.e, for any $n \in \mathbb{N}$,

$E(x_0^n) = 0_B$, *whenever n is odd*
and
$E(x_0^{2n}) = t_{2n} \cdot 1_B$, *where $t_{2n} \in \mathbb{C}$.*

Indeed, for any $n \in \mathbb{N}$,

$$\begin{aligned} E\left(x_0^{2n}\right) &= \sum_{\pi \in NC^{(even)}(2n)} \prod_{(v_1,v_2)\in\pi} K_2^t(x_0, x_0) \\ &= \sum_{\pi \in NC^{(even)}(2n)} \prod_{(v_1,v_2)\in\pi} (t \cdot 1_B) \in \mathbb{C} \cdot 1_B. \end{aligned}$$

Remember that a B-semicircular element is B-even. Hence we only need to consider $NC^{(even)}(2n)$. Also, recall that, moments of B-central elements are multiplicative, like a scalar-valued case. Hence the second equality holds true, in this case. (Note that if x_0 is Not B-central, then we have that

$$E\left(x_0^{2n}\right) = \sum_{\pi \in NC^{(even)}(2n)} \prod_{(v_1,v_2)\in\pi(o)} \widehat{C}(\pi \mid_{(v_1,v_2)} (x_0 \otimes \widehat{C}(\pi \mid_W)(x_0 \otimes x_0)x_0)),$$

where W is an inner block of (v_1, v_2), if exists.)

Now, fix a linear functional $\varphi : A \to \mathbb{C}$. If x_0 is centered for φ and E (i.e $\varphi(x_0) = 0$ and $E(x_0) = 0_B$), then $(A\lg(\{x_0\}) * B,\ \varphi) \subset (A, \varphi)$ is a compatible subalgebra, relative to (A, E). Since x_0 is centered for φ and E, we have that $\varphi(x_0) = \varphi\,(E(x_0))$. Moreover, by the previous observation, $\varphi(x_0^{2n}) = \varphi\left(E(x_0^{2n})\right)$, for $n \in \mathbb{N}$.

2.4. Construction of a compatible NCPSpace (A, φ') related to the given (A, φ) and (A, E).

In this section, we will consider how to construct a NCPSpace, (A, φ'), when a NCPSpace (A, φ) and an amalgamated NCPSpace (A, E), over a unital algebra $B \subset A$ are given. The following proposition makes us possible to construct a new NCPSpace (A, φ'), which is compatible with (A, E), for the given (A, φ).

Proposition 2.13. *Let B be a unital algebra and A, an algebra over B. Suppose that we have a B-functional $E : A \to B$ and a linear functional $\psi : B \to \mathbb{C}$. Then a NCPSpace (A, φ) and a NCPSpace over B, (A, E), are compatible, where $\varphi = \psi \circ E$.*

Proof. Put $\varphi = \psi \circ E : A \to \mathbb{C}$. Then it is trivially a linear functional and hence (A, φ) is a NCPSpace. By definition of φ, we have that

$$\varphi(a) = \psi\,(E(a)) = \psi\left(E^2(a)\right) = \varphi\,(E(a))\,,$$

for all $a \in A$. Therefore, (A, φ) and (A, E) are compatible via $\psi : B \to \mathbb{C}$. ∎

The following theorem shows us that if we have arbitrary NCPSpace (A, φ) and amalgamated NCPSpace (A, E), where $B \subset A$ and $E : A \to B$ is a B-functional, then we can construct a NCPSpace, (A, φ'), which is compatible with the given amalgamated NCPSpace, (A, E).

Theorem 2.14. *Let B be a unital algebra and A, an algebra over B. Suppose that we have a NCPSpace (A, φ) and a NCPSpace over B, (A, E), with its B-functional $E : A \to B$. If we define $\varphi' = \varphi \circ E : A \to \mathbb{C}$, then a new NCPSpace (A, φ') and (A, E) are compatible.*

Proof. Let $\psi = \varphi \mid_B : B \to \mathbb{C}$. Then ψ is a linear functional from B into $\mathbb{C}$. Define a new linear functional,

$$\varphi' = \psi \circ E : A \to \mathbb{C}.$$

Then this new linear functional satisfies that

$$\varphi'(a) = \psi \circ E(a) = \varphi \circ E(a) = \varphi\left(E(a)\right),$$

for all $a \in A$. Therefore, we can get that (A, φ') and (A, E) are compatible. ∎

2.5. **Amalgamated Semicircularity and Semicircularity.**

In this section, we will consider the semicircularity and amalgamated semicircularity under the compatibility.

Definition 2.5. *(1) Let (A, φ) be a NCPSpace. A scalar-valued random variable $a \in (A, \varphi)$ is semicircular, if only the second (scalar-valued) cumulant is nonvanishing. i.e*

$$k_n\left(\underbrace{a, \ldots\ldots, a}_{n-times}\right) = \begin{cases} k_2(a, a) & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We say that a family of random variables ($s \in \mathbb{N}$), $\{a_1, \ldots, a_s\}$, is a semicircular system if each a_j ($j = 1, \ldots, s$) is semicircular and $\{a_1\}, \ldots, \{a_s\}$ are free in (A, φ).

(2) Let B be a unital algebra and (A, E), a NCPSpace over B, with its B-functional $E : A \to B$. A operator-valued random variable $a \in (A, E)$ is B-semicircular (or amalgamated semicircular over B) if only second (operator-valued) cumulant is nonvanishing. i.e

$$K_n\left(\underbrace{a, \ldots\ldots, a}_{n-times}\right) = C^{(n)}\left(a \otimes b_2 a \otimes \ldots \otimes b_n a\right)$$

$$= \begin{cases} C^{(2)}\left(a \otimes b_2 a\right) = K_2(a, a) & \text{if } n = 2 \\ & \\ 0_B & \text{otherwise,} \end{cases}$$

where $b_2, \ldots, b_n \in B$ are arbitrary, for each $n \in \mathbb{N}$ and $\widehat{C} = (C^{(n)})_{n=1}^{\infty} \in I(A, B)$ is the cumulant multiplicative bimodule map induced by E, in the sense of Speicher. We say that a family of B-valued random variables $(s \in \mathbb{N})$, $\{a_1, \ldots, a_s\}$, is a B-semicircular family if each a_j $(j = 1, \ldots, s)$ is B-semicircular and $\{a_1\}, \ldots, \{a_s\}$ are free over B, in (A, E).

Remark 2.3. *The above definitions of semircularity and amalgamated semicircularity are purely combinatorial and purely algebraic. Originally, by Voiculescu, they are defined on a C^*- (or W^*-) algebra framework. Suppose that we have a unital C^*-algebra B and C^*-algebra A containing B, as its C^*-subalgebra. Let $\varphi : A \to \mathbb{C}$ be a state and $E : A \to B$, a suitable conditional expectation. ($\varphi(x^*) = \overline{\varphi(x)} \in \mathbb{C}$ and $E(x^*) = E(x)^* \in B$, for all $x \in A$) Then we have a C^*-probability space (A, φ) and an amalgamated C^*-probability space over B, (A, E). We say that a scalar-valued random variable $a \in (A, \varphi)$ is semicircular if a is self-adjoint and the only nonvanishing cumulant of a is the second one. Also, we say that an operator-valued random variable $a \in (A, E)$ is B-semicircular (or amalgamated semicircular) if a is self-adjoint and the only nonvanishing operaotr-valued cumulant of a is the second one. So, originally, we are very much needed the $*$-structure in both scalar-valued ane operator-valued cases. But, in our definition, we drop this. Of course, we can consider the $*$-structure on our algebras B and A and we can add self-adjointness in our definitions. But, in this Section, we will only consider the combinatorial properties without $*$-structure.*

If a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E), are compatible, there are following semicircular-B-semicircular relation ;

Proposition 2.15. *Let B be a unital algebra and A, an algebra over B. Suppose we have a NCPSpace (A, φ) and a NCPSpace over B, (A, E), with its B-functional $E : A \to B$. Assume that (A, φ) and (A, E) are compatible. Then*

(1) if a B-central B-valued random variable $a \in (A, E)$ is B-semicircular and if $a \in (A, E)$ satisfies the scalar-valued property , then

$$k_n\left(\underbrace{a, ..., a}_{n-times}\right) = 0, \text{ for any odd number } n$$

and

$$k_n\left(\underbrace{a, ..., a}_{2n-times}\right) = \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1,...,v_{2k})\in\pi}\left(\sum_{\theta\in NC^{(even)}(2k)} t^{k(2)}\right)\right)\mu(\pi, 1_{2n}),$$

where $k(2)$ is the number of pair blocks of $\theta \in NC_2(2k)$, only depending on $k \in \mathbb{N}$, for all $n \in \mathbb{N}$. Here, the numer "t" is gotten from

$$K_2^t(a, a) = t \cdot 1_B.$$

(2) if a random variable $a \in (A, \varphi)$ is semicircular and if a is free from B, in (A, φ), then a is B-semicircular, as a B-valued random variable.

Proof. (1) Let $a \in (A, E)$ be a B-semicircular element. Then, by definition,

$$K_n^t\left(\underbrace{a, ..., a}_{n-times}\right) = C^{(n)}\left(\underbrace{a \otimes ..... \otimes a}_{n-times}\right)$$

$$= \begin{cases} C^{(2)}(a, a) = K_2(a, a) & \text{if } n = 2 \\ 0_B & \text{otherwise,} \end{cases}$$

where $\widehat{C} = (C^{(k)})_{k=1}^{\infty} \in I(A, B)$ is the cumulant multiplicative bimodule map induced by E. Assume that $K_2(a, a) = t \cdot 1_B$. Then,

$E(a^n) = 0_B$, if n is odd
and
$$E(a^{2n}) = \sum_{\pi\in NC^{(even)}(2n)} \prod_{(v_1,v_2)\in\pi} K_2(a, a),$$

where
$NC^{(even)}(2n) = \{\theta \in NC(2n) : \theta \text{ does not contain odd blocks}\}$.

(Recall that, by [20], B-semicircularity implies the B-evenness.)

Notice that the above equality holds true, because x_0 is B-central (i.e $x_0 b = b x_0$, $\forall b \in B$). By Section 2.1,

$$k_n\left(\underbrace{a, ..., a}_{n-times}\right) = \sum_{\pi\in NC(n)} \prod_{(v_1,...,v_k)\in\pi} \varphi\left(E(a^k)\right)\mu(\pi, 1_n)$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sum_{\pi \in NC^{(even)}(n)} \prod_{(v_1,...,v_{2k}) \in \pi} \varphi\left(E(a^{2k})\right) \mu(\pi, 1_n) & \text{if } n \text{ is even.} \end{cases}$$

Then, for any $n \in \mathbb{N}$,

$$k_{2n}\left(\underbrace{a, ..., a}_{2n-times}\right) = \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1,...,v_{2k}) \in \pi} \varphi\left(E(a^{2k})\right)\right) \mu(\pi, 1_{2n})$$

$$= \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1,...,v_{2k}) \in \pi} \varphi\left(\sum_{\theta \in NC^{(even)}(2k)} \prod_{(w_1,w_2) \in \theta} K_2(a,a)\right)\right) \mu(\pi, 1_{2n})$$

$$= \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1,...,v_{2k}) \in \pi} \varphi\left(\sum_{\theta \in NC^{(even)}(2k)} \prod_{(w_1,w_2) \in \theta} t\right)\right) \mu(\pi, 1_{2n})$$

$$= \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1,...,v_{2k}) \in \pi} \left(\sum_{\theta \in NC^{(even)}(2k)} t^{|\theta|_2}\right)\right) \mu(\pi, 1_{2n}).$$

(2) Suppose that $a \in (A, \varphi)$ is a semicircular. Since a and B are free in (A, φ), by Section 2.2, we have that

$$K_n^t\left(\underbrace{a, ..., a}_{n-times}\right) = k_n\left(\underbrace{a, ..., a}_{n-times}\right) \cdot 1_B$$

$$= \begin{cases} k_2(a,a) \cdot 1_B & \text{if } n = 2 \\ 0_B & \text{otherwise.} \end{cases}$$

Therefore, a is B-semicircular. ∎

Corollary 2.16. *Let B be a unital algebra and A, an algebra over B. Suppose that (A, φ) is a NCPSpace and (A, E) is a NCPSpace over B, with its B-functional $E : A \to B$. Assume that (A, φ) and (A, E) are compatible. Fix $s \in \mathbb{N}$. If $\{x_1, ..., x_s\}$ is a semicircular system and if $\{x_1, ..., x_s\}$ is free from B, in (A, φ), then $\{x_1, ..., x_s\}$ is a B-semicircular system.*

Proof. By the freeness of $\{x_1, ..., x_s\}$ and B, by (2) of the previous proposition, we have that each x_j $(j = 1, ..., s)$ is B-semicircular, again. Also, by the compatibility, we have vanishing mixed operator-valued cumulant of $x_1, ..., x_s$. Indeed, mixed operator-valued cumulants are ;

$$K_n\left(x_{i_1}, ..., x_{i_n}\right) = C^{(n)}\left(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes ... \otimes b_{i_n} x_{i_n}\right)$$

where $b_{i_2}, ..., b_{i_n} \in B$ are arbitrary and where $\widehat{C} = (C^{(k)})_{k=1}^{\infty} \in I(A, B)$ is the cumulant multiplicative bimodule map induced by E

$$= (\varphi(b_{i_2}) \cdots \varphi(b_{i_n}))\, k_n^t(x_{i_1}, ..., x_{i_n}) \cdot 1_B$$

by the freeness of $\{x_1, ..., x_s\}$ and B, in (A, φ)

$$= (\varphi(b_{i_2}) \cdots \varphi(b_{i_n}))\, (0) \cdot 1_B = 0_B,$$

by the freeness of $\{x_1\}, ..., \{x_s\}$, in (A, φ). Thus $\{x_1, ..., x_s\}$ is a B-semicircular system. ∎

2.6. **Amalgamated Evenness and Evenness.**

In this section, we will consider the evenness and amalgamated evenness under the compatibility.

Definition 2.6. *Let B be a unital algebra and A, an algebra over B. Suppose that we have a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E), with its B-functional $E : A \to B$.*

(1) A random variable $a \in (A, \varphi)$ is even if all odd cumulants of a vanish. i.e

$$k_n \left(\underbrace{a, ..., a}_{n-times} \right) = 0, \text{ for all odd number } n \in \mathbb{N}.$$

(2) An operator-valued random variable $a \in (A, E)$ is B-even (or amalgamated even) if all odd cumulants of a vanish. i.e

$$K_n \left(\underbrace{a, ..., a}_{n-times} \right) = C^{(n)} \left(a \otimes b_2 a \otimes ... \otimes b_n a \right) = 0_B,$$

for all odd number $n \in \mathbb{N}$, where $b_2, ..., b_n \in B$ are arbitrary and $\widehat{C} = (C^{(k)})_{k=1}^{\infty} \in I(A, B)$ is the cumulant multiplicative bimodule map induced by E.

Remark 2.4. *Similar to the semicircular-case, evenness is defined originally on $*$-structure. So, self-adjointness is needed. However, like the previous section, we will only consider the pure combinatorial properties.*

In [20], we observed that ;

Proposition 2.17. *(See [20]) Let (A, E) be a NCPSpace over a unital algebra B. If a B-valued random variable $a \in (A, E)$ is B-even, then*

(1) all odd moments vanish. (The converse is also true by the Möbius inversion.)

(2) we have that, for any $n \in \mathbb{N}$,

$$K_{2n}\left(\underbrace{a, ..., a}_{2n-times}\right) = C^{(2n)}\left(a \otimes b_2 a \otimes ... \otimes b_{2n} a\right)$$
$$= \sum_{\pi \in NC^{(even)}(2n)} \widehat{E}(\pi)\left(a \otimes b_2 a \otimes ... \otimes b_{2n} a\right) \mu(\pi, 1_{2n}),$$

where
$NC^{(even)}(2n) = \{\pi \in NC(2n) : \pi$ *does not contain odd blocks*$\}$

and $b_2, ..., b_n \in B$ are arbitrary. Here, an odd (resp. even) block means a block containing odd (resp. even) number of entries. $\square$

Proposition 2.18. *Let B, (A, φ) and (A, E) be given as before. Assume that (A, φ) and (A, E) are compatible.*

(1) Let $a \in (A, E)$ be even. Then $a \in (A, \varphi)$ is even.

(2) Let $a \in (A, \varphi)$ be even. If a is free from B, in (A, φ), then $a \in (A, E)$ is B-even.

Proof. (1) Let $a \in (A, E)$ be a B-even element. Observe that

$$k_n\left(\underbrace{a, ..., a}_{n-times}\right) = \sum_{\pi \in NC(n)} \left(\prod_{(v_1, ..., v_k) \in \pi} \varphi\left(E(a^k)\right)\right) \mu(\pi, 1_n) = 0,$$

whenever n is odd, since there always exists odd block $(v_1, ..., v_l) \in \pi$, for each $\pi \in NC(n)$. So,

$$k_{2n}\left(\underbrace{a, ..., a}_{2n-times}\right) = \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1, ..., v_{2k}) \in \pi} \varphi\left(E(a^{2k})\right)\right) \mu(\pi, 1_{2n})$$
$$= \sum_{\pi \in NC^{(even)}(2n)} \left(\prod_{(v_1, ..., v_{2k}) \in \pi} \varphi\left(a^{2k}\right)\right) \mu(\pi, 1_{2n}).$$

Therefore, a is (scalar-valued) even, too.

(2) Now, suppose that $a \in (A, \varphi)$ is even and assume that a is free from B, in (A, φ). So, we can get that

$$K_n\left(\underbrace{a, ..., a}_{n-times}\right) = C^{(n)}\left(a \otimes b_2 a \otimes ... \otimes b_n a\right)$$

where $b_2, ..., b_n \in B$ are arbitrary

$$= \left(\varphi(b_2) \cdots \varphi(b_n)\right) k_n^t\left(\underbrace{a, ..., a}_{n-times}\right) \cdot 1_B$$

by the freeness of a and B, in (A, φ)

$$= 0_B,$$

whenever $n \in \mathbb{N}$ is odd, by the evenness of $a \in (A, \varphi)$. Therefore, a is B-even, in (A, E). ∎

Theorem 2.19. *Let B be a unital algebra and A, an algebra over B. Suppose that a NCPSpace (A, φ) and an amalgamated NCPSpace over B, (A, E) are compatible. Let $a \in A$ be an operator and let a linear functional φ be nondegenerated. Then*

a is B-even if and only if a is (scalar-valued) even.

Proof. $(\Rightarrow)$ It is followed from Proposition 2.18, (1).

$(\Leftarrow)$ Assume that $a \in A$ is even. Notice that

[a is B-even]$\overset{def}{\Leftrightarrow}$[all odd B-cumulants vanish]$\overset{(*)}{\Leftrightarrow}$[all odd B-moments vanish].

The (*)-relation holds by the Möbius inversion (See Proposition 2.7, (1)). So, it suffices to show that all odd B-valued moments of a vanish. Trivially, since

$\varphi(x) = \varphi\left(E(x)\right)$, for all $x \in A$,

we have that
$\varphi(a^n) = \varphi\left(E(a^n)\right)$, for all $n \in \mathbb{N}$.

Now, let n be odd. Since all odd moments $\varphi(a^n)$ vanish, all odd B-moments $E(a^n)$ vanish, by the nondegeneratedness of φ. ∎

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